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SELECTING A SUBSET CONTAINING THE BEST ONE OF SEVERAL IFRA POPULATIONS*

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ABSTRACT

Consider $k \geq 2$ increasing failure rate average (IFRA) populations π_i $(i = 1, 2, \ldots, k)$. The IFRA class of populations with nondecreasing failure rate average (FRA) functions $\gamma(t)$ was defined by Birnbaum, Esary and Marshall [Ann. Math. Statist., 37 (1966), 816-825].

We wish to select a subset (of random size) containing the best one of k given populations according to the unknown ordering of $\gamma_i(t)$ $(i = 1, 2, \ldots, k)$ at some fixed value $t = T$. Any selection of a subset which contains the population with the smallest $\gamma(T)$ is regarded as a correct selection (CS).

Three procedures for the above problem are studied. In each case, a preassigned probability P^* is specified and constants are determined so that the procedure R is explicit and satisfies the condition $P\{CS|R\} \geq P^*$ regardless of the true unknown γ-values.

In the first procedure to the problem, we assume that there is no crossover for the k FRA functions $\gamma_i(t)$ for $0 \leq t < \infty$ $(i = 1, 2, \ldots, k)$. We put one unit on test and replace each failed unit by a new one independently distributed with the same life distribution except for a new starting point. The procedure R_1 is: Retain π_i in the selected subset if and only if $N_i + c \leq \frac{N_{min} + c}{d}$, where $c > 0$ and $0 < d \leq 1$ are determined subject to R_1 satisfying the above probability condition; here $N_i = N_i(T)$ is the number of failures observed from π_i by some fixed time T. An asymptotic $(T \to \infty$ and $P^* \to 1)$ solution is obtained and the constants can be obtained from the tables of Milton (University of Minnesota, Department of Statistics, Technical Report No. 27).

In the second procedure to the problem, we allow any crossover for the FRA functions $\gamma_i(t)$ for $0 \leq t < \infty$ $(i = 1, 2, \ldots, k)$. We put a common number N of units on test and no replacement is made of any failed unit. The procedure R_2 is the same as R_1 in form. A small sample binomial solution is obtained. It is shown that asymptotically $(N \to \infty)$ we get the same solution as the one given by procedure R_1 (with fixed T) for k exponential populations.

In the third procedure to the problem, we use the same assumptions as for the first procedure. We put N units on test and wait until all kN units to fail (without any replacement). Let y_i be the total lifetime for N units from π_i $(i = 1, 2, \ldots, k)$. Procedure R_3 is: Retain π_i in the selected subset if and only if $y_i > b \cdot Y_{max}$, where $b > 0$ is determined subject to R_3 satisfying the basic probability requirement. An asymptotic solution is obtained and the constant needed can be obtained from tables of Milton which are referred to above.

In all the three procedures studied for our problem, we do not require the knowledge of the particular form of the IFRA populations; in fact the k populations need not even be of the same functional form. For each formulation it is proved that the proposed procedure has a monotonicity property, namely that for any two populations the better population has higher probability of having put in the selected subset.

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CHAPTER 1.

Introduction.

Given $k \geq 2$ increasing failure rate average (IFRA) populations, we wish to select a subset (of random size) containing the best one of the k populations on the basis of a common number of observations from each of the k populations. The selection criteria makes use of the failure rate average (FRA) functions of the populations under consideration. Except in Chapter 3, it is assumed that no pair of these functions crossover. Then the population with the smallest FRA is considered as the best population. The appropriateness of this criteria will be clear once we define the class of IFRA distributions.

If the failure distribution $F(t)$ has a density $f(t)$, then the failure rate function $r(t)$ is defined by

$$(1.1) \qquad r(t) = \frac{f(t)}{1 - F(t)} \quad ;$$

clearly this is defined only for those values of t for which $F(t) < 1$. Then, $F(t)$ is called an increasing failure rate (IFR) distribution if and only if $r(t)$ is nondecreasing in t. We shall use some of the properties of IFR distribution pointed out and proved by Barlow and Proschan [1].

The FRA function $\gamma(t)$ is defined as

$$(1.2) \qquad \gamma(t) = \frac{1}{t}\int_0^t r(x)dx.$$

Following Birnbaum, Esary and Marshall [3], $F(t)$ is called an IFRA distribution if and only if $\gamma(t)$ is nondecreasing in t. Using (1.1) and (1.2) we obtain the failure rate average FRA function in the form

$$(1.3) \qquad \gamma(t) = -\frac{1}{t}\log[1 - F(t)].$$

By differentiating $\gamma(t)$ with respect to t, we see that if $F(t)$ is IFR, then it is also IFRA but the converse is not true. A counterexample is given by Barlow and Proschan [2]; thus IFRA is a wider class of distributions. Many widely-used distributions like the exponential, gamma, Weibull and truncated normal distributions are members of the IFR class and hence also belong to the IFRA class. Although there is a natural analog to $r(t)$ for discrete distributions, we shall be interested here only in the continuous IFRA populations $F_i(t)$ with $t \geq 0$ and $F_i(0) = 0$ $(i = 1, 2, \ldots, k)$.

In many situations, where the nature of physical devices is to wear out in time, it is reasonable to assume that the underlying life distribution is IFRA. Since the FRA function $\gamma(t)$ in (1.3) represents a failure rate, it seems logical to use it as a selection criterion. In the later chapters we use either $F(t)$ or $F(t|\gamma)$ or $F^{(1)}(T|\gamma)$ to denote an IFRA population whose FRA function is $\gamma = \gamma(t)$.

Three procedures for handling our goal of selecting a subset containing the best population are considered. In each case we use the same probability condition; this enables us to determine certain constants which make the procedure explicit. The solutions obtained are general in the sense that the particular form of the IFRA distributions is not used and does not have to be known. In fact, the k populations need not be of the same functional form.

Problem 1 (see Chapter 2).

For the k given IFRA populations, it is assumed that there is no crossover for the k FRA functions $\gamma_i(t)$ for $0 \leq t < \infty$ $(i = 1, 2, \ldots, k)$; the one with the smallest $\gamma_i(t)$ is then called a best population. In this problem we put one unit on test from each population and replace each failed unit by a brand new item independently distributed with the same life-time distribution except for the new starting point. The proposed procedure R_1 is based on the number of failures in a fixed time T. An asymptotic ($T \to \infty$ and the preassigned probability P^* close to one) solution is obtained and the constants needed for the procedure can be obtained from the tables of Milton [8] or Gupta [5] as a function of P^* and k.

Problem 2 (see Chapter 3).

For the k given IFRA populations we allow the FRA functions $\gamma_i(t)$ to crossover for $0 \leq t < \infty$. The best population is defined as the one having the smallest $\gamma(T)$; T is fixed. For each population a common number N of units are put on test for a common time T and no replacement is made of any failed unit. A proposed procedure R_2 is based on the number of failures in time T. The small sample binomial solution does not depend on the functional form of the IFRA populations and these need not be given. It is found that asymptotically $(N \to \infty)$ the procedure R_2 yields the same solution as procedure R_1 with k exponential populations; the constants needed are tabulated for some cases.

Problem 3 (see Chapter 4).

Under the same assumptions as in Problem 1, a procedure R_3 is proposed which requires that a common number N of units from each of the k populations be put on test and we wait for all kN units to fail (without any replacement). The total lifetime for all N units from the same population is observed. An asymptotic $(N \to \infty)$ solution is obtained and the constant needed can be obtained from tables of Milton [8] or Gupta [5].

CHAPTER 2.

2.1 Formulation of the Problem.

Let π_i denote an IFRA population whose FRA function is $\gamma_i(t)$ $(i = 1, 2, \ldots, k)$. It is assumed that there is no crossover for any pair of the k functions $\gamma_i(t)$ for $0 \leq t < \infty$. Let the ordered values of the $\gamma_i = \gamma_i(t)$ for any t be denoted by

$$(1.1) \qquad \gamma_{[1]} \leq \gamma_{[2]} \leq \cdots \leq \gamma_{[k]};$$

the ordering is the same for all $t \geq 0$. It is assumed that there is no a priori information available about the correct pairing of the ordered $\gamma_{[i]}$ and the k given populations.

Any population whose FRA function is $\gamma_{[1]}$ is called a 'best' population. The goal is to select a nonempty subset of the k populations containing a best population. A correct selection (CS) is defined as a selection of any subset of the k given populations which contains at least one best population and we use $P\{CS|R_1\}$ to denote the probability of correct selection using a procedure R_1.

The problem is to find a procedure R_1 such that for preassigned probability P^* (with $\frac{1}{k} < P^* < 1$) we satisfy the requirement

$$(1.2) \qquad P\{CS|R_1\} \geq P^*$$

regardless of the true unknown $\underset{\sim}{\gamma} = (\gamma_{[1]}, \gamma_{[2]}, \ldots, \gamma_{[k]}) \in \Gamma$; here Γ is a set of functions which satisfies our above assumption on the absence of crossovers and for any fixed $t \geq 0$ forms a k-tuple with nonnegative elements.

2.2 Proposed Procedure R_1.

For each population π_i $(i = 1, 2, \ldots, k)$, we put one unit on test and replace each failed unit immediately by a brand new unit independently distributed with the same life time distribution except for the new starting point.

Let $N_i = N_i(T)$ be the random number of failures from π_i in the fixed time T; later in the paper we assume that T is large in order for large sample theory to hold. $N_i(T)$ is frequently referred to as the renewal counting process. Let the k ordered integer values of the N_i be given by

$$(2.1) \qquad N_{[1]} \leq N_{[2]} \leq \cdots \leq N_{[k]}.$$

In terms of these we define the

Procedure R_1:

"Retain population π_i in the selected subset if and only if

$$(2.2) \qquad N_i + c \leq \frac{N_{[1]} + c}{d} \quad ;$$

here c and d are nonnegative constants to be determined with $c > 0$ and $0 < d \leq 1$."

The constants c and d are chosen to satisfy the basic probability requirement (1.2).

In many of the ranking and selection problems, it is found that only one constant in the proposed procedure is sufficient to specify the procedure. However in this formulation we need to find a pair of values (c, d). It is shown in Section 2.8 that using the procedure in (2.2) with $c = 0$, we cannot in general find a d-value to satisfy the P^*-requirement (1.2). For any fixed $c > 0$, there is a unique d-value satisfying (1.2), since as will be shown later the $P\{CS|R_1\}$ for fixed $c > 0$ decreases with d and approaches one as d approaches zero.

2.3 $P\{CS|R_1\}$ and its infimum over Γ.

Let $N_i(T)$ $(i = 1, 2, \ldots, k)$ be as defined in Section (2.2) and let $N_{(i)} = N_{(i)}(T)$ correspond to a population $\pi_{(i)}$ whose γ-value is $\gamma_{[i]}$. Suppose the failures for $\pi_{(i)}$ occur at random times denoted by

$$X_1^{(i)},\ X_1^{(i)} + X_2^{(i)}, \ldots,\ X_1^{(i)} + X_2^{(i)} + \ldots + X_{N_{(i)}}^{(i)}, \quad (X_j^{(i)} \geq 0)$$

so that X_j denotes the time between the $(j-1)\underline{st}$ and $j\underline{th}$ failures. The X_j are independent and have the same IFRA distribution. Note that for any integer $m \geq 0$ and any i

$$(3.1)\qquad P\{N_{(i)} = m\} = P\{\sum_{j=1}^{m} X_j^{(i)} \leq T \text{ and } \sum_{j=1}^{m+1} X_j^{(i)} > T|\gamma_{[i]}\}$$

$$= F^{(m)}(T|\gamma_{[i]}) - F^{(m+1)}(T|\gamma_{[i]}),$$

where

$$(3.2)\qquad F^{(m)}(T|\gamma_{[i]}) = P\{N_{(i)} \geq m\} = P\{\sum_{j=1}^{m} X_j^{(i)} \leq T|\gamma_{[i]}\}$$

and, by definition, $F^{(0)}(T|\gamma_{[i]})$ is identically one for all T, $\underset{\sim}{\gamma}$.

Now using (3.1), (3.2) and the definition of the procedure R_1, we have, writing $P\{CS|R_1\}$ as $P\{CS|c, d\}$ with R_1 understood,

$$(3.3)\qquad P\{CS|c, d\} = P\{N_{(1)} + c \leq \frac{N_{[1]}+ c}{d}\}$$

$$= P\{N_{(j)} \geq (N_{(1)}+ c)d - c, \quad j = 2, 3, \ldots, k\}$$

$$= \sum_{\alpha=0}^{\infty} P\{N_{(1)} = \alpha\} \cdot P\{N_{(j)} \geq [[(\alpha + c)d - c]], \; j = 2, 3, \ldots, k\}$$

$$= \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma_{[1]}) - F^{(\alpha+1)}(T|\gamma_{[1]})\} \prod_{j=2}^{k} \{F^{(\beta)}(T|\gamma_{[j]})\},$$

where $\beta = [[(\alpha + c)d - c]]$ denotes the smallest nonnegative integer not less than $(\alpha + c)d - c$.

<u>Theorem 2.3.1.</u>

(a) $P\{CS|c, d\}$ is an increasing function of c for fixed d.

(b) $P\{CS|c, d\}$ is a decreasing function of d for fixed c.

Proof of (a):

For any pair (c_1, c_2) with $c_1 < c_2$, letting $\beta_i = [[(\alpha + c_i)d - c_i]]$ $(i = 1,2)$, we have $\beta_1 \geq \beta_2$ and

$$F^{(\beta_1)}(T|\gamma_{[j]}) = P\{N_{(j)} \geq \beta_1\} \leq P\{N_{(j)} \geq \beta_2\} = F^{(\beta_2)}(T|\gamma_{[j]}). \tag{3.4}$$

Combining the inequalities in (3.3) for each j with the help of (3.4) gives

$$P\{CS|c_1, d\} < P\{CS|c_2, d\}.$$

Proof of (b):

For any pair (d_1, d_2) with $d_1 < d_2$, letting $\beta_i' = [[(\alpha + c)d_i - c]]$ $(i = 1, 2)$, we have $\beta_1' \leq \beta_2'$ and hence by applying (3.4) to (3.3)

$$P\{CS|c, d_1\} > P\{CS|c, d_2\}.$$

2.3.1 Infimum of $P\{CS|c, d\}$ over Γ.

To find the infimum of $P\{CS|c, d\}$, we shall need the following results.

Lemma 2.3.1.

For $i < j$ and any finite T,

$$F(t|\gamma_{[i]}) \leq F(t|\gamma_{[j]}) \qquad (0 \leq t \leq T). \tag{3.5}$$

Proof:

Using the definition (1.3) of Chapter 1 for $\gamma = \gamma(t)$, we can write $F(t|\gamma_{[i]})$ as

$$F(t|\gamma_{[i]}) = 1 - e^{-\gamma_{[i]}t}. \tag{3.6}$$

Since $F(t|\gamma)$ is an increasing function of γ, the result holds for any $t > 0$ and in particular for $t \leq T$.

Lemma 2.3.2.

If $F(t)$, $G(t)$ are any two cdf's with $F(0) = G(0) = 0$ and $F(t) \leq G(t)$ for $0 \leq t \leq T$, then for any integer $m \geq 1$

$$F^{(m)}(t) \leq G^{(m)}(t) \qquad (0 \leq t \leq T). \tag{3.7}$$

Proof:

The result is true for $m = 1$ by hypothesis. Supposing it is true for $m-1$, we prove it by induction on m. Using the definition of $F^{(m)}(t)$ and the induction hypothesis, we have for any t in the interval $[0, T]$

$$(3.8) \qquad F^{(m)}(t) = P\{\sum_{i=1}^{m} X_i \leq t\} = \int_0^t F^{(m-1)}(t - x)dF(x) \leq \int_0^t G^{(m-1)}(t - x)dF(x).$$

Integrating the rhs of (3.8) by parts and using the hypothesis we have for any t in the interval $[0, T]$

$$\begin{aligned} F^{(m)}(t) &\leq - \int_0^t F(x)dG^{(m-1)}(t - x) \\ &\leq - \int_0^t G(x)dG^{(m-1)}(t - x) \\ &= \int_0^t G(t - x)dG^{(m-1)}(x) = G^{(m)}(t), \end{aligned}$$

which proves the result.

Since the two cdf's in (3.5) form a pair satisfying (3.7) we obtain from Lemmas 2.3.1 and 2.3.2 the result

<u>Lemma 2.3.3</u>.

If $\gamma_{[i]}(t) \leq \gamma_{[j]}(t)$ for $0 \leq t \leq T$ and for any pair $i, j \leq k$, then for any integer $m \geq 1$ and any finite T

$$(3.9) \qquad F^{(m)}(t|\gamma_{[i]}) \leq F^{(m)}(t|\gamma_{[j]}) \qquad (0 \leq t \leq T).$$

In particular, we make use of (3.9) for $t = T$.

Using the following theorem, we find that the least favorable configuration of γ-values in Γ is the one for which $\gamma_{[k]} = \gamma_{[k-1]} = \ldots = \gamma_{[1]} = \gamma$ (say), for all $t \geq 0$ where the common function $\gamma \geq 0$ is not yet determined. The following theorem is now proved in a somewhat more general form so that we can also use it later. Using a development quite similar to that in (3.3) we find that the probability $A_i = A_i(\underset{\sim}{\gamma})$ of including in the selected subset the population associated with $\gamma_{[i]}$ is

$$A_i = \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma_{[i]}) - F^{(\alpha+1)}(T|\gamma_{[i]})\} \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta)}(T|\gamma_{[j]}), \tag{3.10}$$

where β is the integer defined after (3.3).

<u>Theorem 2.3.2</u>.

(a) For each i $(i = 1, 2, \ldots, k)$, A_i is a nonincreasing function of $\gamma_{[i]}$ holding all $\gamma_{[j]}$ $(j \neq i)$ fixed, i.e., for $\gamma_{[i]}$ in the interval $(\gamma_{[i-1]}, \gamma_{[i+1]})$.

(b) For each j $(j = 1, 2, \ldots, k, j \neq i)$, A_i is a nondecreasing function of $\gamma_{[j]}$, holding all $\gamma_{[h]}$ $(h \neq i, h \neq j)$ fixed.

Proof of (a):

From (3.10) we have

$$A_i = \sum_{\alpha=0}^{\infty} F^{(\alpha)}(T|\gamma_{[i]}) \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta)}(T|\gamma_{[j]}) - \sum_{\alpha=0}^{\infty} F^{(\alpha+1)}(T|\gamma_{[i]}) \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta)}(T|\gamma_{[j]}). \tag{3.11}$$

For $\alpha = 0$ we find that $\beta = 0$ and hence the first term T_1 on rhs of (3.11) can be written as

$$T_1 = 1 + \sum_{\alpha=1}^{\infty} F^{(\alpha)}(T|\gamma_{[i]}) \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta)}(T|\gamma_{[j]}).$$

The second term T_2 on rhs of (3.11) can be written as

$$T_2 = - \sum_{\alpha=1}^{\infty} F^{(\alpha)}(T|\gamma_{[i]}) \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta_1)}(T|\gamma_{[j]}),$$

where $\beta_1 = [[(\alpha - 1 + c)d - c]]$. Combining T_1 and T_2 we can write (3.11) as

$$A_i = 1 - \sum_{\alpha=1}^{\infty} F^{(\alpha)}(T|\gamma_{[i]}) \{ \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta_1)}(T|\gamma_{[j]}) - \prod_{\substack{j=1 \\ j \neq i}}^{k} F^{(\beta)}(T|\gamma_{[j]}) \}. \tag{3.12}$$

Since $\beta_1 \leq \beta$, the difference inside the braces in (3.12) is always nonnegative for each α. Let $\gamma_1 = \gamma_1(t)$ be such that $\gamma_{[i]} \leq \gamma_1 \leq \gamma_{[i+1]}$ for all $t \geq 0$.

Holding $\gamma_{[j]}$ $(j = 1, 2,\ldots, k,\ j \neq i)$ fixed, using the Lemma 2.3.2 for each α, we obtain

$$(3.13) \qquad A_i \geq 1 - \sum_{\alpha=1}^{\infty} F^{(\alpha)}(T|\gamma_1)\{\prod_{\substack{j=1\\ j\neq i}}^{k} F^{(\beta_1)}(T|\gamma_{[j]}) - \prod_{\substack{j=1\\ j\neq i}}^{k} F^{(\beta)}(T|\gamma_{[j]})\}.$$

Retracing the above steps from (3.13) back to (3.10) we obtain

$$(3.14) \qquad A_i \geq \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma_1) - F^{(\alpha+1)}(T|\gamma_1)\} \prod_{\substack{j=1\\ j\neq i}}^{k} \{F^{(\beta)}(T|\gamma_{[j]})\},$$

which proves part (a).

Proof of part (b):

From Lemma 2.3.3 we know that for each α, $F^{(\beta)}(T|\gamma_{[j]})$ is a nondecreasing function of $\gamma_{[j]}$ $(j = 1, 2,\ldots, k)$. Hence holding all $\gamma_{[h]}$ $(h \neq i)$ fixed and using the Lemma 2.3.3 for each j, we get the required result.

If we now put $i = 1$ in (3.10), then (3.10) and (3.3) are the same and using part (b) of Theorem 2.3.3, we find that the first step in obtaining the least favorable configuration is to set $\gamma_{[k]} = \gamma_{[k-1]} = \ldots = \gamma_{[1]} = \gamma$ (say) for all $t \geq 0$. Hence we obtain

$$(3.15) \qquad P\{CS|c, d\} \geq \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma) - F^{(\alpha+1)}(T|\gamma)\}\{F^{(\beta)}(T|\gamma)\}^{k-1} = A(\gamma), \text{ (say)}.$$

It should be pointed out that all the results upto (3.15) also hold for any arbitrary failure distributions $F_i(t)$ $(i = 1, 2,\ldots, k)$. But further results of this chapter depend on the special properties enjoyed only by IFRA populations.

We also note that this lower bound to the $P\{CS|c, d\}$ is obtained for fixed T; later we shall let $T \to \infty$. Before obtaining an asymptotic expression for $A(\gamma)$ we shall first prove some properties of IFRA distributions; Barlow and Proschan [1] have shown these properties to hold for IFR distributions. The proofs for the results given below are similar to those of [1]. For convenience, we write $\overline{F}(t) = 1 - F(t)$.

2.4 Some Properties of the IFRA Distribution.

Lemma 2.4.1.

If $F(t)$ is IFRA, then $[\overline{F}(t)]^{\frac{1}{t}}$ is nonincreasing in t.

Proof:

Using the definition of $\gamma(t)$ in (1.3) of Chapter 1 we write for $t > 0$

$$(4.1) \qquad [\overline{F}(t)]^{\frac{1}{t}} = e^{-\gamma(t)}.$$

Since $\gamma(t)$ is nondecreasing in t, the result follows.

Lemma 2.4.2.

Any IFRA distribution has finite moments of all orders.

Proof:

Using Lemma 2.4.1, we find that for $t \geq 0$

$$(4.2) \qquad \overline{F}(x) \leq [\overline{F}(t)]^{\frac{x}{t}} \qquad \text{for } x > t.$$

Since $F(t)$ is a cdf there must exist a $t_0 > 0$ such that $\overline{F}(t) < 1$ for $t \geq t_0$ amd $\gamma(t) > 0$ for $t \geq t_0$. For any $r \geq 0$ and $t \geq t_0$

$$(4.3) \qquad \int_{t_0}^{\infty} x^r \,\overline{F}(x)dx \leq \int_{t_0}^{\infty} x^r [\overline{F}(t_0)]^{\frac{x}{t_0}} dx$$

$$= \int_{t_0}^{\infty} x^r e^{-\gamma(t_0)x} dx$$

$$\leq \frac{\Gamma(r+1)}{[\gamma(t_0)]^{r+1}} < \infty.$$

It follows easily from (4.3) that

$$(4.4) \qquad (r+1)\int_0^{\infty} x^r \,\overline{F}(x)dx < \infty.$$

An integration by parts gives

$$(4.5) \qquad (r+1)\int_0^{\infty} x^r \,\overline{F}(x)dx = \int_0^{\infty} x^{r+1} dF(x) + \lim_{x\to\infty} x^{r+1}\,\overline{F}(x).$$

From the fact that both terms on the rhs of (4.5) are nonnegative and the lhs of (4.5) is finite, it follows that each term on the rhs is finite. In particular, the $(r + 1)$st moment exists.

Theorem 2.4.1.

If (a) $F(t)$ is IFRA with mean μ, and (b) $G(t) = 1 - e^{-t/\mu}$ is an exponential distribution, and (c) $\varphi(t)$ is any continuous increasing function of t then

$$\int_0^\infty \varphi(t)\overline{F}(t)dt \leq \int_0^\infty \varphi(t)\overline{G}(t)dt. \tag{4.6}$$

Proof:

Since the IFRA distribution $F(t)$ and the exponential distribution $G(t)$ have a common mean μ we claim that $\overline{F}(t)$ crosses $\overline{G}(t)$ exactly once from above, say at t_0. To prove this we proceed as follows.

If $\gamma_1(t)$ and $\gamma_2(t)$ denote the FRA of $F(t)$ and $G(t)$, respectively, then clearly $\gamma_2(t) = \frac{1}{\mu}$. We can assume that $F(t)$ is not exponential since, if it is, the result is trivial. If $\gamma_1(t)$ and the horizontal line at $\frac{1}{\mu}$ cross, it has to be only once, say at t_0, so that $\gamma_1(t) < \frac{1}{\mu}$ for $t < t_0$ and $\gamma_1(t) > \frac{1}{\mu}$ for $t > t_0$. Hence, using the definition of $\gamma(t)$ in (1.3) of Chapter 1 we have

$$\overline{F}(t) > \overline{G}(t) \qquad \text{for} \qquad t < t_0 \tag{4.7a}$$

and

$$\overline{F}(t) < \overline{G}(t) \qquad \text{for} \qquad t > t_0. \tag{4.7b}$$

If $\gamma_1(t)$ and the horizontal line at $\frac{1}{\mu}$ do not cross then either $\gamma_1(t) < \frac{1}{\mu}$ for all t or $\gamma_1(t) > \frac{1}{\mu}$ for all t. Hence, using the definition of $\gamma_1(t)$, if $\gamma_1(t) < \frac{1}{\mu}$ for all t then

$$\overline{F}(t) > \overline{G}(t) \qquad \text{for all } t \tag{4.8}$$

and a similar result holds for $\gamma_1(t) > \frac{1}{\mu}$ for all t. However, from (4.6) and (4.8)

$$\mu = \int_0^\infty \overline{F}(t)dt > \int_0^\infty \overline{G}(t)dt = \mu,$$

a contradiction; a similar contradiction holds if $\gamma_1(t) > \frac{1}{\mu}$ for all t. Now we can write

$$(4.9) \qquad \int_0^\infty \varphi(t)\overline{F}(t)dt - \int_0^\infty \varphi(t)\overline{G}(t)dt = \int_0^\infty [\varphi(t) - \varphi(t_0)][\overline{F}(t) - \overline{G}(t)]dt$$
$$= \int_0^{t_0} [\varphi(t) - \varphi(t_0)][\overline{F}(t) - \overline{G}(t)]dt$$
$$+ \int_{t_0}^\infty [\varphi(t) - \varphi(t_0)][\overline{F}(t) - \overline{G}(t)]dt.$$

Since $\varphi(t) < \varphi(t_0)$ and $\overline{F}(t) > \overline{G}(t)$ for $t < t_0$ and $\varphi(t) > \varphi(t_0)$ and $\overline{F}(t) < \overline{G}(t)$ for $t > t_0$, the rhs of (4.9) is nonpositive, which proves the result.

Corollary.

If $F(t)$ is IFRA with rth moment μ_r then for $r \geq 0$

$$(4.10) \qquad \mu_r \leq r!\mu_1^r.$$

Proof:

The proof is simple and is exactly the same as in [1] for the IFR distribution. However we give it here for completeness.

For $r \geq 1$, let $\varphi(t) = t^{r-1}$ in the above, then it follows from (4.7b) that $\lim_{t\to\infty} t^r \overline{F}(t) = 0$ and hence $0 < \mu_r = r\int_0^\infty t^{r-1}\overline{F}(t)dt$. Using Theorem 2.4.1 it follows that $\mu_r \leq r\int_0^\infty t_0^{r-1}\overline{G}(t)dt = r!\,\mu_1^r$. Since $\mu_1 = \mu$ and letting σ^2 denote variance of $F(t)$, we obtain from (4.10) for $r = 2$

$$(4.11) \qquad \sigma = \sqrt{\mu_2 - \mu_1^2} \leq \mu.$$

2.5 Asymptotic $(T\to\infty)$ Lower Bound to $P\{CS|c, d\}$.

It is known (see [4], [9]) that the renewal counting process $N(T|\gamma)$ is asymptotically $(T\to\infty)$ normally distributed with asymptotic mean T/μ and asymptotic variance $T\sigma^2/\mu^3$, where $\mu > 0$ and σ^2 are the mean and variance of the IFRA distribution F.

Recalling the definition of the procedure R_1 we can write $A(\gamma)$, defined in (3.15), as

$$(5.1) \qquad A(\gamma) = P\{N_i(T|\gamma) \geq [[N_1(T|\gamma) + c]]d - c, \ i = 2, 3, \ldots, k\}$$

where the $N_i(T|\gamma)$ $(i = 1, 2, \ldots, k)$ are independent and identically distributed random variables. Asymptotically $(T \to \infty)$ we note that $A = A(\gamma)$ depends only on $\theta = \frac{1}{\mu} > 0$ and the variance σ^2 and using the above results we obtain

$$(5.2) \qquad A = P\{\frac{N_i(T) - \theta T}{\sigma\theta\sqrt{\theta T}} \geq \frac{[N_1(T) - \theta T]d}{\sigma\theta\sqrt{\theta T}} - \frac{(c + \theta T)(1 - d)}{\sigma\theta\sqrt{\theta T}}, \ i = 2, 3, \ldots, k\}$$

$$\approx \int_{-\infty}^{\infty} [1 - \Phi\{xd - \frac{(c + \theta T)(1 - d)}{\sigma\theta\sqrt{\theta T}}\}]^{k-1} d\Phi(x),$$

where $\Phi(x)$ is the standard normal cdf. Since we know from (4.11) that for the IFRA distributions $0 \leq \sigma\theta \leq 1$, we get for large T the inequality

$$(5.3) \qquad A \geq \int_{-\infty}^{\infty} [1 - \Phi\{xd - \frac{(c + \theta T)(1 - d)}{\sqrt{\theta T}}\}]^{k-1} d\Phi(x)$$

$$= \int_{-\infty}^{\infty} \Phi^{k-1}[xd + \frac{(c + \theta T)(1 - d)}{\sqrt{\theta T}}] d\Phi(x).$$

We note that the rhs of (5.3) is a function of θT only. Hence in order to minimize the rhs of (5.3), we should minimize $\frac{(c + \theta T)}{\sqrt{\theta T}}$ with respect to θ. We find that it is minimized when $\theta T = c$ and the minimum value is $2\sqrt{c}$. [It is easy to see that when $\theta T \to \infty$, the rhs of (5.3) approaches one and the same conclusion also holds when θT approaches zero; hence $\theta T = c$ yields a minimum value.]

Using the inequalities (3.15) and (5.3) with the above result, we obtain for large T

$$(5.4) \qquad P\{CS|c, d\} \geq \int_{-\infty}^{\infty} \Phi^{k-1}[xd + 2(1 - d)\sqrt{c}\,] d\Phi(x).$$

It is important to note that the rhs of (5.4) no longer depends on T or θ

although we are assuming that T is large. It is also interesting to note that we have attained a minimum here without letting θ approach an extreme value.

Hence the remaining problem is to solve for (c, d) the equation

$$(5.5) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}[xd + 2(1-d)\sqrt{c}\,]d\Phi(x) = P^*.$$

We should note that if only P^* is specified, then we cannot solve the equation (5.5) for both c and d. In fact there will be many pairs (c, d) which satisfy the equation (5.5). However if P^* and a suitable d were given, we could then solve the equation (5.5) for c. Note that we can rewrite equation (5.5) as

$$(5.6) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}\left[\frac{x\sqrt{\rho} + \mu}{\sqrt{1-\rho}}\right]d\Phi(x) = P^*,$$

where

$$(5.7) \qquad \rho = \frac{d^2}{(1 + d^2)} \quad \text{and} \quad H = \frac{2(1-d)\sqrt{c}}{\sqrt{1+d^2}} .$$

Now we can use tables of Milton [8] or Gupta [5] by entering with the known ρ value to obtain the tabulated H-value from which we get the solution in c as $c = \dfrac{H^2(1 + d^2)}{4(1 - d)^2}$.

For the special case $k = 2$, it is easily seen that (5.5) reduces to

$$(5.8) \qquad \Phi\left[\frac{2(1-d)\sqrt{c}}{\sqrt{1+d^2}}\right] = P^*,$$

which requires only the usual normal table to find c for a given value of d.

In the next section we shall apply another condition on the expected size of the selected subset and the two conditions together can be used to evaluate both c and d simultaneously.

2.6 Expected Size of the Selected Subset.

For the procedure R_1, the size S of the selected subset is a chance variable which can take on only integer values from 1 to k, inclusive. For any fixed values of T, k and P^*, the expected size of the selected subset will be a function of the true configuration $\underset{\sim}{\gamma}$. We shall denote it by $E(S|c, d)$.

To find an exact expression for the expected size of the subset, we note that S can be written as

$$(6.1) \qquad S = \sum_{i=1}^{k} Z_i,$$

where $Z_i = 1$ if π_i is included in the selected subset and $Z_i = 0$ otherwise. Then using (3.10)

$$(6.2) \qquad E(S|c, d) = \sum_{i=1}^{k} E(Z_i)$$

$$= \sum_{i=1}^{k} P\{N_{(i)} + C \leq \frac{N_{[1]} + c}{d}\}$$

$$= \sum_{i=1}^{k} \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma_{[i]}) - F^{(\alpha+1)}(T|\gamma_{[i]})\} \prod_{\substack{j=1 \\ j \neq i}}^{k} \{F^{(\beta)}(T|\gamma_{[j]})\},$$

where $\beta = [[(\alpha + c)d - c]]$ denotes the smallest nonnegative integer not less than $(\alpha + c)d - c$.

2.6.1 <u>An additional condition to determine a unique pair (c, d)</u>.

As remarked earlier, we need an additional condition on c and d so that we can solve the equation (5.5) for both c and d. The required condition is that the pair (c, d) satisfying (5.5) minimizes $E(S|c, d)$ in the configuration given by

$$(6.3) \qquad \gamma_{[k]}(t) = \gamma_{[k-1]}(t) = \ldots = \gamma_{[2]}(t) \quad \text{for all} \quad t \geq 0, \; \gamma_{[2]}(T) = \delta(T), \; \gamma_{[1]}(T) = 0,$$

where $\delta(T)$ is a positive constant which need not be specified to find the pair (c, d). In fact no knowledge of the functional form of the IFRA distributions is required. In the following $F(T|\delta)$ will denote the value of the cdf at $t = T$ when the population has a FRA given by $\gamma_{[2]}(t)$.

Under the configuration given by (6.3) we can use (6.2) to write $E(S|c, d)$ as

$$(6.4) \qquad E(s|c, d) = \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|0) - F^{(\alpha+1)}(T|0)\}\{F^{(\beta)}(T|\delta)\}^{k-1}$$

$$+ (k - 1) \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\delta) - F^{(\alpha+1)}(T|\delta)\}\{F^{(\beta)}(T|\delta)\}^{k-2}F^{(\beta)}(T|0).$$

By (3.1) and (3.2), we can write $\{F^{(\alpha)}(T|0) - F^{(\alpha+1)}(T|0)\} = P\{N(T|0) = \alpha\}$ and $F^{(\beta)}(T|0) = P\{N(T|0) \geq (\alpha + c)d - c\}$. Since $N(T|0) = 0$ and since $\beta = 0$ when $\alpha = 0$, we can write (6.4) as

$$(6.5) \qquad E(S|c, d) = 1 + (k-1) \sum_{\alpha=0}^{\nu} \{F^{(\alpha)}(T|\delta) - F^{(\alpha+1)}(T|\delta)\}$$

$$= 1 + (k - 1)\, P\{N(T|\delta) \leq \nu\}$$

where ν can be taken as $\frac{c(1 - d)}{d}$ in the final expression of (6.5).

Now we note that the rhs of (6.5) is an increasing function of ν. Hence the required condition calls for a pair (c, d) satisfying (6.5) which gives the minimum ν-value. We shall now show explicitly how this can be done.

We now rewrite (5.5) in the form

$$(6.6) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}[xd + 2\sqrt{\nu d(1 - d)}\,]\, d\Phi(x) = P^{*}.$$

Note that for any fixed d the lhs is an increasing function of ν and approaches one as $\nu \to \infty$. Hence it follows that for any fixed d, the equation (6.6) can be solved for ν. The problem is then to find a d for which the resulting ν is minimum. We shall now, for convenience use the form (5.6) rather than (6.6) so that we can use (5.7).

Note that we can write $\nu = \frac{H^2(1 + d^2)}{4d(1 - d)}$ and the problem is to find a d which minimizes ν and satisfies (6.6). Differentiating $\frac{H^2(1 + d^2)}{d(1 - d)}$ with respect to d and equating to zero we get

$$(6.8) \qquad d(1 - d)[2d\, H^2 + 2(1 + d^2)H \frac{\partial H}{\partial d}\,] = H^2(1 + d^2)(1 - 2d)$$

from which we get

$$(6.9) \qquad (d^2 + 2d - 1) = - \frac{2d(1 - d)(1 + d^2)}{H} \frac{\partial H}{\partial d}.$$

We observe from tables of Milton [8] and Gupta [5] that H varies slightly with d. Hence as a first approximation we take H to be constant so that the second term in (6.9) is zero. Then the resulting equation has one positive root given by

$$d = \sqrt{2} - 1 \approx .414. \tag{6.10}$$

Note that for the case $k = 2$, H is a constant and the required d is exactly equal to $\sqrt{2} - 1$.

However, in order to find an improved value of d, we should find $\frac{\partial H}{\partial d}$.

Differentiating both sides of (5.6) with respect to ρ, we get

$$\begin{aligned} &\int_{-\infty}^{\infty} (x + H\sqrt{\rho})\Phi^{k-2}\left(\frac{x\sqrt{\rho} + H}{\sqrt{1-\rho}} \right)\varphi\left(\frac{x\sqrt{\rho} + H}{\sqrt{1-\rho}} \right)\varphi(x)dx \\ &= -2(1-\rho)\sqrt{\rho}\ \frac{\partial H}{\partial \rho} \int_{-\infty}^{\infty} \Phi^{k-2}\left(\frac{x\sqrt{\rho} + H}{\sqrt{1-\rho}} \right)\varphi\left(\frac{x\sqrt{\rho} + H}{\sqrt{1-\rho}} \right)\varphi(x)dx, \end{aligned} \tag{6.11}$$

where $\varphi(x)$ is the standard normal d.f. We now write

$$\varphi\left(\frac{x\sqrt{\rho} + H}{\sqrt{1-\rho}} \right)\varphi(x) = \varphi(H)\varphi\left(\frac{x + H\sqrt{\rho}}{\sqrt{1-\rho}} \right) \tag{6.12}$$

and using the transformation

$$y = \frac{x + H\sqrt{\rho}}{\sqrt{1-\rho}}$$

we write (6.12) as

$$\begin{aligned} &\int_{-\infty}^{\infty} y\ \Phi^{k-2}(y\sqrt{\rho} + H\sqrt{1-\rho})\varphi(y)dy \\ &= -2\sqrt{\rho(1-\rho)}\ \frac{\partial H}{\partial \rho} \int_{-\infty}^{\infty} \Phi^{k-2}(y\sqrt{\rho} + H\sqrt{1-\rho})\varphi(y)dy. \end{aligned} \tag{6.13}$$

Integrating by parts, the lhs of (6.13) can be written as

$$\sqrt{\rho}\ (k-2) \int_{-\infty}^{\infty} \varphi(y)\Phi^{k-3}(y\sqrt{\rho} + H\sqrt{1-\rho})\varphi(y\sqrt{\rho} + H\sqrt{1-\rho})dy. \tag{6.14}$$

Also we can write

$$(6.15)\qquad \varphi(y)\varphi(y\sqrt{\rho} + H\sqrt{1-\rho}) = \varphi\left(H\sqrt{\frac{1-\rho}{1+\rho}}\right)\varphi\left(y\sqrt{1+\rho} + H\sqrt{\frac{\rho(1-\rho)}{1+\rho}}\right).$$

Now using the transformation

$$(6.16)\qquad z = y\sqrt{1+\rho} + H\sqrt{\frac{\rho(1-\rho)}{1+\rho}}$$

in (6.14), we can write (6.15) as

$$(6.17)\qquad \frac{(k-2)}{\sqrt{1+\rho}}\ \varphi\left(H\sqrt{\frac{1-\rho}{1+\rho}}\right)\int_{-\infty}^{\infty}\Phi^{k-3}\left(\frac{z\sqrt{\rho(1+\rho)} + H\sqrt{1-\rho}}{1+\rho}\right)\varphi(z)dz$$

$$= -2\sqrt{1-\rho}\ \frac{\partial H}{\partial\rho}\int_{-\infty}^{\infty}\Phi^{k-2}(y\sqrt{\rho} + H\sqrt{1-\rho})\varphi(y)dy.$$

Note that (6.17) can be solved for $\frac{\partial H}{\partial \rho}$ and the values of the integrals in (6.17) can be found from the tables of Milton [8] or Gupta [5].

Since $\rho = \dfrac{d^2}{1+d^2}$ we get

$$(6.18)\qquad \frac{\partial H}{\partial d} = \frac{\partial H}{\partial \rho}\cdot\frac{\partial \rho}{\partial d} = \frac{\partial H}{\partial \rho}\cdot\frac{2d}{(1+d^2)^2}\ ,$$

where $\frac{\partial H}{\partial \rho}$ is obtained from (6.17). We note that $\frac{\partial H}{\partial \rho}$ will be negative in sign.

Let ϵ denote the rhs of (6.9). Then

$$(6.19)\qquad \epsilon = -\frac{4d^2(1-d)}{H(1+d^2)}\cdot\frac{\partial H}{\partial \rho} > 0.$$

We can now use a recursive scheme to get an improved value of d which satisfies (6.9).

As a starting value we take $d_0 = .414$ and find ρ_0 using (5.7). For given P^* and the k value we now find H_0 value from the above-mentioned tables. Using these d_0, ρ_0, H_0 and k values in (6.19) we find ϵ_0. We then write the equation (6.9) as

$$(6.20)\qquad d^2 + 2d - (1+\epsilon_0) = 0.$$

The solution of the equation (6.20) is

$$d_1 = -1 + \sqrt{2+\epsilon_0}.$$

We now take d_1 as the next starting value and repeat the above procedure to get ϵ_1 and using (6.20) with ϵ_1 replacing ϵ_0 to get d_2. The procedure can be repeated until the d-value converges to the true d value which satisfies (6.9). It is found numerically, as illustrated below, that the convergence is very rapid if we start with $d_0 = \sqrt{2} - 1$

Case (1) $P^* = .90$, $k = 5$

$$d_0 = .414,\ \rho_0 = .1463,\ H_0 = 1.9254;\ \frac{\partial H_0}{\partial \rho} = -.1526,\ \epsilon_0 = .0272$$

$$d_1 = .423,\ \rho_1 = .1518,\ H_1 = 1.9246;\ \frac{\partial H_1}{\partial \rho} = -.1372,\ \epsilon_1 = .0250$$

$$d_2 = .423.$$

Case (2) $P^* = .99$, $k = 5$

$$d_0 = .414,\ \rho_0 = .1463,\ H_0 = 2.8028;\ \frac{\partial H_0}{\partial \rho} = -.03154,\ \epsilon_0 = .0039$$

$$d_1 = .415.$$

The following table gives the d-value which minimizes ν for selected value of P^* and k. The c-value corresponding to d and ν is also included.

Table

P^*	k	d	ν	c
.90	2	.414	1.9777	1.397
	3	.420	3.1759	2.300
	5	.423	4.4733	3.329
.95	2	.414	3.2664	2.308
	3	.420	4.5833	3.319
	5	.414	5.9684	4.217
.99	2	.414	6.5534	4.630
	3	.414	7.9930	5.647
	5	.414	9.4827	6.690

2.6.2 A Secondary Problem.

To use the asymptotic $(T \to \infty)$ results obtained in Section 2.5 numerically we need to know how large T has to be before the results can be used adaquately.

One way of handling this problem is considered in this section.

We note that the equation (5.3) depends on T only through the quantity θT, where θ is the reciprocal of the mean μ. Hence if we assume that an upper bound $\bar{\theta}$ on θ can be given or approximated, then by the discussion after (5.3) we set $\bar{\theta} T_0 = c_0$ (where c_0 is the unique solution in c given by (5.5)); this gives a lower bound T_0 on T, i.e.,

$$(6.21) \qquad T > \frac{c_0}{\bar{\theta}} .$$

One way to improve the adequacy of the normal approximation is to set P^* (sufficiently) close to one. This will increase c_0 (since $d_0 \approx .414$ is now fixed) and hence by (6.21) will increase the lower bound on the fixed time T of the experiment.

2.7 A Special Case: Exponential Populations.

Suppose for small T we are interested in selecting a subset containing the best one of k exponential populations. We know that the FRA function of an exponential distribution is a constant given by $\gamma(t) = \theta$ for all $t > 0$. It is also known that if the failures occur according to an exponential distribution with FRA function $\gamma = \gamma(t)$, then the renewal counting process $N(T|\gamma)$ is a Poisson process.

As a first step to obtain the inf $P\{CS|c, d\}$ we use the configuration $\gamma_{[k]} = \gamma_{[k-1]} = \ldots = \gamma_{[1]}$ for all t and the result obtained in (3.15).

Using the definition in (3.2) for exponential case, the rhs of (3.15) takes the form

$$(7.1) \qquad P\{CS|c, d\} \geq \sum_{\alpha=0}^{\infty} e^{-\theta T} \frac{(\theta T)^{\alpha}}{\alpha!} \left\{ \sum_{j=\beta}^{\infty} e^{-\theta T} \frac{(\theta T)^{j}}{j!} \right\}^{k-1}.$$

Let $B(\theta T)$ denote the rhs of (7.1). We will now show that there exists a finite $\theta T > 0$ which minimizes $B(\theta T)$.

We first note that $\beta = 0$ when $\alpha = 0$. Writing the first term of $B(\theta T)$ separately, we have

$$(7.2)\qquad B(\theta T) = e^{-\theta T} + \sum_{\alpha=1}^{\infty} e^{-\theta T} \frac{(\theta T)^{\alpha}}{\alpha!} \{\sum_{j=\beta}^{\infty} e^{-\theta T} \frac{(\theta T)^{j}}{j!}\}^{k-1}.$$

Since $0 \leq B(\theta T) \leq 1$ for all $\theta T \geq 0$, it follows from (7.2) that $B(0) = 1$.

Now for large θT, we note that a normal approximation to $B(\theta T)$ can be applied and is given by the rhs of (5.3). Now as $\theta T \to \infty$ we easily notice that the rhs of (5.3) approaches one. Since $B(\theta T)$ is continuous in θT, there must exist a $\theta_0 (\theta_0 > 0)$ at which $B(\theta T)$ assumes the minimum value. It is conjectured that θ_0 is unique; numerical calculations indicate that it is unique.

The remaining problem is to solve for (c, d) the equation

$$(7.3)\qquad \sum_{\alpha=0}^{\infty} e^{-\theta_0} \frac{\theta_0^{\alpha}}{\alpha!} \{\sum_{j=\beta}^{\infty} e^{-\theta_0} \frac{\theta_0^{j}}{j!}\}^{k-1} = P^*,$$

where β is given after (3.3). For given P^*, k and a suitable d (or c) we can solve the equation (7.3) for c (or d). This is accomplished by using a table that evaluates the lhs of (7.3) for given c, d and k. Table I does exactly this for selected values of c, d and k; here θT values in steps of 0.1 were used in order to locate θ_0.

We now make numerical comparisons between the large sample theory of Section 2.6 and the small sample theory of this section.

We notice from numerical calculations that as $P^* \to 1$, the d-value needed to minimize ν (and hence the expected size of the subset) according to small sample theory approaches the d-value $(d = \sqrt{2} - 1)$ obtained by large sample theory. With P^* in the range .90-.97, which is generally used, the d-value needed to minimize ν is approximately .2. The following table (for the selected values of P^* and k) illustrates this remark.

Table

P^*	k	c	d	ν
.99	5	2	.20	8.0
		5	.41	7.2
		10	.52	9.2
.99	2	2	.30	4.6
		3	.40	4.5
		10	.60	6.6
.965	5	1	.20	4.0
		2	.30	4.6
		10	.60	6.6
.955	2	.0.5	.20	2.0
		2	.44	2.4
		5	.59	3.7

2.8 Upper Bound on P^* for the Procedure R_1 (with $c = 0$).

A remark was made in Section 2.2 that the two constants c and d in the definition of the procedure R_1 are necessary in order to solve the problem. In other words, when $c = 0$ in the procedure R_1 we cannot, in general, find a d-value which satisfies P^*-condition (1.2). In the following we shall find an upper bound for P^* when $c = 0$ and $\gamma(T)$ is finite.

When $c = 0$, $A(\gamma)$ as defined in (3.15) reduces to

$$(8.1) \qquad A(\gamma) = \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma) - F^{(\alpha+1)}(T|\gamma)\}\{F^{(\beta^*)}(T|\gamma)\}^{k-1}$$

$$= 1 - F(T|\gamma) + \sum_{\alpha=1}^{\infty} \{F^{(\alpha)}(T|\gamma) - F^{(\alpha+1)}(T|\gamma)\}\{F^{(\beta^*)}(T|\gamma)\}^{k-1},$$

where $\beta^* = [[\alpha d]]$; the latter is zero for $\alpha = 0$.

We know from the definition in (3.2) that for any finite $\gamma(T)$, $F^{(\alpha)}(T|\gamma)$ is a decreasing function of α and approaches zero as $\alpha \to \infty$. Now as $d \to \infty$ through positive values

$$(8.2) \qquad A(\gamma) \to 1 - F(T|\gamma) + F(T|\gamma)\{F^{(1)}(T|\gamma)\}^{k-1} = 1 - F(T|\gamma) + F^k(T|\gamma)$$

$$= e^{-\gamma(T)T} + (1 - e^{-\gamma(T)T})^k,$$

where the relation (1.2) of Chapter 1 is used.

Now considering $\gamma(T)$ for fixed T as an unknown and differentiating the rhs of (8.2) with respect to $\gamma(T)$ and equating it to zero, we find that the value of $\gamma(T)$ which minimizes (8.2) is the solution of

$$(8.3) \qquad 1 - e^{-\gamma_0(T)T} = \left(\frac{1}{k}\right)^{\frac{1}{k-1}}.$$

Hence because of the P^*-condition,

$$(8.4) \qquad P^* = A(\gamma_0) \to 1 - \left(\frac{1}{k}\right)^{\frac{1}{k-1}} + \left(\frac{1}{k}\right)^{\frac{k}{k-1}}.$$

Notice that $P^* < 1$ for $k \geq 2$. Now because of part (b) of Theorem 2.3.1 we see that when $c = 0$, no matter what d is chosen $P^* < 1$ and hence the problem in general cannot be solved. But by a proper choice of c we can see that the rhs of (8.1) with β replacing β^* will approach one as $d \to 0$. Hence the problem can be solved. The following small table gives the upper bound $\bar{P}^*$ of P^* (when $c = 0$) for a selected value of k.

Table

k	$\bar{P}^*$
2	.750
3	.615
4	.527
5	.466

2.9 Monotonicity Property of the Procedure R_1.

Let A_i $(i = 1, 2, \ldots, k)$ denote the probability of including in the selected subset a population whose FRA function is $\gamma_{[i]}$.

Theorem 2.8.1.

$A_i \geq A_h$ for $i \leq h$, $i, h = 1, 2, \ldots, k$.

Proof:

It suffices to prove $A_i \geq A_{i+1}$. Using the definition of A_i in (3.10) and the part (a) of Theorem 2.3.2 with $\gamma_{[j]}$ $(j \neq i)$ fixed we obtain

$$(9.1) \qquad A_i \geq \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma_{[i+1]}) - F^{(\alpha+1)}(T|\gamma_{[i+1]})\}\{\prod_{\substack{j=1 \\ j\neq i,i+1}}^{k} F^{(\beta)}(T|\gamma_{[j]})\}F^{(\beta)}(T|\gamma_{[i+1]}).$$

By Lemma 2.3.3 we know that $F^{(\beta)}(T|\gamma_{[i]}) \leq F^{(\beta)}(T|\gamma_{[i+1]})$ for every β.

Hence we get the inequality

$$(9.2) \qquad A_i \geq \sum_{\alpha=0}^{\infty} \{F^{(\alpha)}(T|\gamma_{[i+1]}) - F^{(\alpha+1)}(T|\gamma_{[i+1]})\} \prod_{\substack{j=1 \\ j\neq i+1}}^{k} F^{(\beta)}(T|\gamma_{[j]}) = A_{i+1}.$$

CHAPTER 3.

In this chapter we shall let the problem remain basically the same as in Chapter 1. The only change we make is that we now allow the k FRA functions $\gamma_i(t)$ to crossover for $0 \leq t < \infty$. The best population is then defined as a population whose γ-value at $t = T$ is the smallest. As before $T > 0$ is specified and fixed. In this chapter we shall write $\gamma_{[i]}$ for $\gamma_{[i]}(T)$ so that ordering of γ_i values is at $t = T$. We propose a procedure R_2 which is based on binomial sampling. It should be pointed out that the problem of ranking binomial populations was first solved by Gupta and Sobel [7]. However, the procedure used here is different from that used in [7] and as a result the asymptotic analysis is also different. It should also be noted that although we assume IFRA populations in the development, we do not use this property and hence that assumption is not needed in this chapter.

3.1 Proposed Procedure R_2.

For each of k IFRA populations π_i $(i = 1, 2,..., k)$, a common number N of units are put on test for a common time T without replacing any failures. Let N_i denote the number of failures observed from π_i. Let X_i denote the life time of a unit from π_i. Then the probability of failure of any unit from π_i, which we denote as p_i, is

$$(1.1) \qquad p_i = F(T|\gamma_i) = p\{X_i \leq T|\gamma_i\} = 1 - e^{-\gamma_i(T)T}.$$

Note that the ordering of $\gamma_i = \gamma_i(T)$ is the same as the ordering of the p_i values.

Let the ordered values of N_i be given by

$$(1.2) \qquad N_{[1]} \leq N_{[2]} \leq \cdots \leq N_{[k]}.$$

In terms of these we define

Procedure R_2.

"Retain population π_i in the selected subset if and only if

$$(1.3) \qquad N_i + c \le \frac{N_{[1]} + c}{d} \; ;$$

here c and d are nonnegative constants with $c > 0$ and $0 < d \le 1$." The remarks concerning c and d given after (2.2) also apply here.

3.2 $P\{CS|R_2\}$ and its infimum over Γ.

Let $N_{(i)}$ $(i = 1, 2, \ldots, k)$ correspond to a population $\pi_{(i)}$ whose γ-value is $\gamma_{[i]}$. We note that $N_{(i)}$ has a binomial distribution with parameters N and $p_{[i]}$. Now using the definition of procedure R_2, and the same argument as for Section 2.3

$$(2.1) \qquad P\{CS|R_2\} = P\{N_{(1)} + c \le \frac{N_{[1]} + c}{d}\}$$
$$= \sum_{\alpha=0}^{N} \{\binom{N}{\alpha} p_{[1]}^{\alpha}(1 - p_{[1]})^{N-\alpha}\} \prod_{i=2}^{k} \{\sum_{j=\beta}^{N} \binom{N}{j} p_{[i]}^{j}(1 - p_{[i]})^{N-j}\},$$

where $\beta = [[(\alpha + c)d - c]]$ is again the smallest nonnegative integer not less than $(\alpha + c)d - c$.

Using the relation between a cumulative binomial distribution and the beta distribution, we can write for $\beta \ge 1$

$$(2.2) \qquad \sum_{j=\beta}^{N} \binom{N}{j} p_{[i]}^{j}(1 - p_{[i]})^{N-j} = \frac{N!}{(\beta - 1)!(N - \beta)!} \int_{0}^{p_{[i]}} y^{\beta-1}(1 - y)^{N-\beta} dy;$$

for $\beta = 0$ the lhs of (2.2) is identically one.

We note that the rhs of (2.2) is an increasing function of $p_{[i]}$ $(i = 1,2,\ldots, k)$. Holding $\gamma_{[j]}$ $(j \neq i)$ fixed and using this result for each $i \ge 2$ in (2.1), we can add the results for each α and obtain

$$(2.3) \qquad P\{CS|R_2\} \ge \sum_{\alpha=0}^{N} \binom{N}{\alpha} p^{\alpha}(1 - p)^{N-\alpha}\{\sum_{j=\beta}^{N} \binom{N}{j} p^{j}(1 - p)^{N-j}\}^{k-1},$$

where $p_{[k]} = p_{[k-1]} = \ldots = p_{[1]} = p$ (say). We note that if the p's are equal then by (1.1) the γ's are also, i.e.,

$$(2.4) \qquad \gamma_{[k]} = \gamma_{[k-1]} = \ldots = \gamma_{[1]} = \gamma \quad \text{(say)}.$$

Let $B_1(p)$ denote the rhs of (2.3). We will now show that there exists a p $(0 < p < 1)$ which minimizes $B_1(p)$.

We first note from the definition of β that $\beta = 0$ when $\alpha = 0$. Writing the first term of $B_1(p)$ separately, we have

$$(2.5) \qquad B_1(p) = (1 - p)^N + \sum_{\alpha=1}^{N} \binom{N}{\alpha} p^\alpha (1 - p)^{N-\alpha} \{ \sum_{j=\beta}^{N} \binom{N}{j} p^j (1 - p)^{N-j} \}^{k-1}.$$

Since $0 \leq B_1(p) \leq 1$ for all $0 \leq p \leq 1$ it follows easily from (2.5) that $B_1(0) = 1$.

Starting with the last term $(\alpha = N)$, $B_1(p)$ can be written as

$$(2.6) \qquad B_1(p) = p^{kN} + p^N \{ \sum_{j=\beta}^{N-1} \binom{N}{j} p^j (1 - p)^{N-j} \}^{k-1}$$
$$+ \sum_{\alpha=0}^{N-1} \{ \binom{N}{\alpha} p^\alpha (1 - p)^{N-\alpha} \} \{ \sum_{j=\beta}^{N} \binom{N}{j} p^j (1 - p)^{N-j} \}^{k-1}.$$

It follows from (2.6) that $B_1(1) = 1$. Thus $B_1(p)$ achieves its maximum value (one) at $p = 0$ and also at $p = 1$. Since $B_1(p)$ is continuous in p there must exist a p_0 $(0 < p_0 < 1)$ at which $B_1(p)$ assumes its minimum value. It is conjectured that p_0 is unique; numerical calculations indicate that it is unique.

Hence the problem is to solve for pairs (c, d) the equation

$$(2.7) \qquad \sum_{\alpha=0}^{N} \{ \binom{N}{\alpha} p_0^\alpha (1 - p_0)^{N-\alpha} \} \{ \sum_{j=\beta}^{N} \binom{N}{j} p_0^j (1 - p_0)^{N-j} \}^{k-1} = P^*,$$

where p_0 now is a function of c, d, k and N.

Given P^*, k, N and a value of d (or c) we can use trial and error methods to find a c (or d) value and the resulting p_0 such that (2.7) holds. We have not tried to set an additional condition as in Chapter 2 to determine c and d simultaneously, although that approach could also be applied. A small table is provided at the end of Section 3.3 which gives the p_0 value as well as the resulting minimum value of the lhs of (2.7) for selected values of c, d, k and N.

3.3 Asymptotic $(N \to \infty)$ Lower Bound for $P\{CS|R_2\}$.

By the definition of the procedure R_2, we can write $B_1(p)$, the rhs of (2.3), as

$$(3.1) \qquad B_1(p) = P\{N_i \geq (N_k + c)d - c,\ i = 1, 2,\ldots, k-1\},$$

where the N_i are independent and identically distributed binomial random variables. For fixed p and large N we can use the central limit theorem to get a normal approximation to $B_1(p)$, namely

$$(3.2) \qquad B_1(p) \approx \int_{-\infty}^{\infty} \Phi^{k-1}\left(xd + \frac{(Np + c)(1 - d)}{\sqrt{Npq}}\right)d\Phi(x),$$

where $q = 1 - p$. By straightforward differentiation we find that the rhs of (3.2) is minimized at $p_0 = \frac{c}{N + 2c}$. Hence as $N \to \infty$, the minimum value $p_0 \to 0$. Since in our problem we equate P^* to the minimum of $B_1(p)$, it is clear that we have not shown that (3.2) is a valid asymptotic $(N \to \infty)$ normal approximation.

On the basis of numerical calculations we conjecture that as $N \to \infty$, Np_0 approaches a finite positive constant λ_0, where p_0 is defined as the value of p which minimizes $B_1(p)$. Under this conjecture, we prove the following lemma.

Lemma 3.3.1.

As $N \to \infty$ and $Np_0 \to \lambda_0$

$$(3.3) \qquad \sum_{\alpha=0}^{N} \{\tbinom{N}{\alpha}p_0^{\alpha}(1 - p_0)^{N-\alpha}\}\{\sum_{j=\beta}^{N} \tbinom{N}{j}p_0^{j}(1 - p_0)^{N-j}\}^{k-1}$$
$$\to \sum_{\alpha=0}^{\infty} e^{-\lambda_0}\frac{\lambda_0^{\alpha}}{\alpha!}\{\sum_{j=\beta}^{\infty} e^{-\lambda_0}\frac{\lambda_0^{j}}{j!}\}^{k-1}.$$

Proof:

It is well-known that as $N \to \infty$ and $Np_0 \to \lambda_0$

$$(3.4) \qquad \tbinom{N}{\alpha}p_0^{\alpha}(1 - p_0)^{N-\alpha} \to e^{-\lambda_0}\frac{\lambda_0^{\alpha}}{\alpha!} \quad \text{for each } \alpha.$$

Clearly for each α

$$(3.5)\qquad \sum_{j=\beta}^{N} \binom{N}{j} p_0^j (1 - p_0)^{N-j} = 1 - \sum_{j=0}^{\beta-1} \binom{N}{j} p_0^j (1 - p_0)^{N-j}$$

$$\to 1 - \sum_{j=0}^{\beta-1} e^{-\lambda_0} \frac{\lambda_0^j}{j!} = \sum_{j=\beta}^{\infty} e^{-\lambda_0} \frac{\lambda_0^j}{j!} .$$

Hence for each α

$$(3.6)\qquad \binom{N}{\alpha} p_0^{\alpha} (1 - p_0)^{N-\alpha} \{ \sum_{j=\beta}^{N} \binom{N}{j} p_0^j (1 - p_0)^{N-j} \}^{k-1} \to e^{-\lambda_0} \frac{\lambda_0^{\alpha}}{\alpha!} \{ \sum_{j=\beta}^{\infty} e^{-\lambda_0} \frac{\lambda_0^j}{j!} \}^{k-1} .$$

Now let $g_N(\alpha)$ denote the left member and $g(\alpha)$ denote the right member of (3.6). Then for any fixed positive integer $N_0 \leq N$ we can write

$$(3.7)\qquad |\sum_{\alpha=0}^{N} g_N(\alpha) - \sum_{\alpha=0}^{\infty} g(\alpha)| \leq | \sum_{\alpha=0}^{N_0} g_N(\alpha) - \sum_{\alpha=0}^{N_0} g(\alpha)| + | \sum_{\alpha=N_0+1}^{N} g_N(\alpha)| + | \sum_{\alpha=N_0+1}^{\infty} g(\alpha)| .$$

Since

$$(3.8)\qquad \sum_{\alpha=N_0+1}^{N} g_N(\alpha) \leq \sum_{\alpha=N_0+1}^{N} \binom{N}{\alpha} p_0^{\alpha} (1 - p_0)^{N-\alpha} ,$$

the right hand sum can be made uniformly close to the corresponding sum of Poisson terms with parameter λ_0. We now define N_0, for given $\varepsilon > 0$, so that

$$(3.9)\qquad \sum_{\alpha=N_0+1}^{\infty} e^{-\lambda_0} \frac{\lambda_0^{\alpha}}{\alpha!} \leq \frac{\varepsilon}{3} .$$

It then follows that

$$(3.10)\qquad \sum_{\alpha=N_0+1}^{\infty} g(\alpha) \leq \frac{\varepsilon}{3} .$$

Combining the results of (3.6), (3.8), (3.9) and (3.10) we have from (3.7), for N sufficiently large (and greater than N_0)

$$(3.11)\qquad |\sum_{\alpha=0}^{N} g_N(\alpha) - \sum_{\alpha=0}^{\infty} g(\alpha)| \leq \varepsilon$$

which proves the lemma.

Hence for large N, the asymptotic solution to the main problem is given by the equation

$$(3.12) \qquad \sum_{\alpha=0}^{\infty} e^{-\lambda_0} \frac{\lambda_0^{\alpha}}{\alpha!} \left\{ \sum_{j=\beta}^{\infty} e^{-\lambda_0} \frac{\lambda_0^{\alpha}}{\alpha!} \right\}^{k-1} = P^*.$$

We notice that this equation is exactly the same as equation (7.3) of Chapter 2, with λ_0 equal to θ_0. Hence for given P^*, k and a suitable d (or c) we can use Table I to solve for c (or d). This will be the asymptotic solution.

The procedure R_2 also has the monotonicity property like the procedure R_1. The proof of this is exactly similar to the proof given in Section 2.9 and is omitted here.

The following table gives the exact value, $B_1(p_0)$, of the lhs of equation (2.7) for some selected values of k, c, d and N. It also gives the corresponding asymptotic $(N \to \infty)$ value, $B(\lambda_0)$, for the lhs of equation (3.12).

Table

k	c	d	N	p_0	Value* of λ_0	$B_1(p_0)$	$B(\lambda_0)$
2	1	.3	10	.18	2.0	.96277	.95174
			50	.04		.95401	
			100	.02		.95291	
3	1	.3	10	.18	2.1	.93114	.91155
			50	.04		.91569	
			100	.02		.91379	
2	1	.6	10	.08	.8	.73908	.73531
			50	.02		.73868	
3	1	.6	10	.08	1.0	.60391	.59374
			50	.02		.59478	
2	3	.4	10	.27	3.8	.99543	.98819
			50	.07		.99019	
3	3	.4	10	.27	3.9	.99107	.97736
			50	.07		.98161	

* λ_0 is the value that minimizes the lhs of (3.12). The fact that $Np_0 \to \lambda_0$ as $N \to \infty$ for any fixed triples (k, c, d) considered is clearly indicated by this table; this result has not been proved however. It has been shown that the value of $B_1(p_0) \to B(\lambda_0)$ as $N \to \infty$.

CHAPTER 4.

In this chapter, we propose a different procedure R_3 for the same problem (with the same assumptions) as described in Chapter 2. A common number N of units from each population are put on test; the procedure R_3 requires that we wait for all kN units to fail (without any replacement) and observe the total life-time for all N units from the same population. The procedure is the same as one used by Gupta [6] for gamma populations and our asymptotic $(N \to \infty)$ results for IFRA populations are also similar to his corresponding results for exponential populations. No knowledge of the particular form of the IFRA populations is required, but the property of IFRA populations is used in this chapter.

4.1 Proposed Procedure R_3.

From each IFRA population π_i $(i = 1, 2, \ldots, k)$ we take N observations X_{ij} $(j = 1, 2, \ldots, N)$ and compute $y_i = \sum_{j=1}^{N} X_{ij}$. Let the ordered values of the k observed y_i be given by

$$Y_{[1]} \le Y_{[2]} \le \ldots \le Y_{[k]}. \tag{1.1}$$

In terms of these we define

Procedure R_3.

"Retain population π_i in the selected subset if and only if

$$Y_i \ge b \cdot Y_{[k]}, \tag{1.2}$$

where b is a constant with $0 < b \le 1$ and is chosen to be the smallest member which satisfies the P^*-condition (1.2) of Chapter 2 for all $\underset{\sim}{\gamma} \in \Gamma$."

It is clear that such a smallest number must exist since for $b = 0$ we include all populations in the selected subset.

4.2 $P\{CS|R_3\}$ and its Infimum Over Γ.

Let $Y_{(i)}$ be the random variable which corresponds to a population whose

γ-value is $\gamma_{[k-i+1]}$ $(i = 1, 2,\ldots, k)$. Now using (3.1), (3.2) of Chapter 2 and the definition of the procedure R_3, we have letting $P\{CS|R_3\} = P\{CS\ |b\}$ with R_3 understood,

$$(2.2)\qquad P\{CS|b\} = P\{Y_{(k)} \geq b\cdot Y_{[k]}\}$$
$$= P\{y_{(j)} \leq \frac{Y_{(k)}}{b}\ ,\ j = 1, 2,\ldots, k\text{-}1\}$$
$$= \int_0^{\infty} \prod_{j=2}^{k} F^{(N)}(\tfrac{t}{b}|\gamma_{[j]})d\ F^{(N)}(t|\gamma_{[1]}).$$

By definition $\gamma_{[j]} = \gamma_{[j]}(t)$ and by assumption $\gamma_{[j]} \geq \gamma_{[1]}$ for all $t > 0$, $j = 2, 3,\ldots, k$. Hence using Lemma 2.3.3 for each j separately we find that the first step in obtaining the least favorable configuration is to set

$$(2.3)\qquad \gamma_{[k]} = \gamma_{[k-1]} =\ldots= \gamma_{[1]} = \gamma \text{ (say)} \quad \text{for all} \quad t > 0$$

in (2.2) and we get

$$(2.4)\qquad P\{CS|b\} \geq \int_0^{\infty} \{F^{(N)}(\tfrac{t}{b}|\gamma)\}^{k-1}\cdot\ d\ F^{(N)}(t|\gamma).$$

4.3 Asymptotic Lower Bound to $P\{CS|b\}$.

Since an IFRA population has finite moments of all orders, we can use the central limit theorem for large N and we find that the random variable $Y_1 = Y$ (say), defined in Section 4.1, is asymptotically $(N \to \infty)$ normally distributed with mean $N\mu$ and variance $N\sigma^2$, where μ and σ^2 are the mean and variance of the underlying IFRA distribution. Recalling the definition of procedure R_3 and using this asymptotic property, we write (2.4) in the form

$$(3.1)\qquad P\{CS|b\} \geq P\{Y_i \leq \frac{Y_{(k)}}{b}\ ,\ i = 1, 2,\ldots, k\text{-}1\},$$

where Y_j $(j = 1, 2,\ldots, k)$ are independent identically distributed random variables. We can write (3.1) as

$$(3.2) \qquad P\{CS|b\} \geq P\{ \frac{Y_i - N\mu}{\sigma\sqrt{N}} \leq \frac{Y_{(k)} - N\mu}{b\sigma\sqrt{N}} + \frac{\mu(1-b)\sqrt{N}}{b\sigma}, \; i = 1, 2, \ldots, k-1\}$$

$$\approx \int_{-\infty}^{\infty} \Phi^{k-1}(\frac{t}{b} + \frac{\mu(1-b)\sqrt{N}}{b\sigma}) \, d\,\Phi(t).$$

For any IFRA population we know by (4.11) of Chapter 2 that $\sigma \leq \mu$ and hence we have the conservative asymptotic result

$$(3.3) \qquad P\{CS|b\} \geq \int_{-\infty}^{\infty} \Phi^{k-1}(\frac{t}{b} + \frac{(1-b)\sqrt{N}}{b}) \, d\,\Phi(t).$$

Since the rhs of (3.3) is free of parameter the remaining problem is to solve for b the equation

$$(3.4) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}(\frac{t}{b} + \frac{(1-b)\sqrt{N}}{b}) \, d\,\Phi(t) = P^*.$$

We can write (3.4) as

$$(3.5) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}(\frac{t\sqrt{\rho} + H}{\sqrt{1-\rho}}) \, d\,\Phi(t) = P^*,$$

where

$$(3.6) \qquad \rho = \frac{1}{1+b^2} \quad \text{and} \quad H = \frac{(1-b)\sqrt{N}}{\sqrt{1+b^2}}$$

so that tables of Milton [8] or Gupta [5] can be used. However, it should be noted that these tables can be used to find H given ρ, P^* and k or to find P^* given ρ, H and k. Since both ρ and H are functions of b, it follows that for $k > 2$ the b-value cannot be found directly and may need trial and error methods. For $k = 2$, the equation (3.5) reduces to

$$(3.7) \qquad \Phi(H) = P^*$$

and using regular normal tables, the equation can be solved for b given P^* and N.

As remarked earlier the procedure R_3 is the same as the one used by Gupta [6] for gamma populations. We find that the rhs of (3.3) also holds asymptotically $(N \to \infty)$ for exponential populations. Gupta (see eq. (4.10) of [6]) has given a different asymptotic form in place of the rhs of (3.3). When his r is set equal to 1, the two asymptotic forms are comparable and indeed asymptotically equivalent.

It is found numerically, as shown in the selected cases below, that his exact solution for $r = 1$ (i.e., the exponential case) and the solution for our IFRA problem obtained by using (3.5) are approximately the same. Given P^*, k and N we first find an approximately b-value from the tables in [6]. Using this approximating b-value (and the same k and N) we use (3.6) to find the lhs of (3.5) from tables in [8] or [5]. This already turns out to be very close to P^*. Hence we can use the tables in [6] (for the special case $r = 1$) to get approximate solutions for our problem also.

If we let P_1^* denote the resulting left side of (3.5) we can form a recursion by using $P^* + (P^* - P_1^*) = 2P^* - P_1^*$ as a new starting value for entering the tables in [6]. For example, if $k = 4$, $N = 16$, $P^* = .95$ then we find that $P_1^* = .94$ and $2P^* - P_1^* = .96$ gives a new b-value of $.456$ and a $P_2^* = .946$. This method gives adequate results after a few recursions. For another example if $k = 3$, $N = 16$, $P^* = .95$ then we find that $P_1^* = .942$ and $2P^* - P_1^* = .958$ gives a new b-value $.484$ and a $P_2^* = .952$. For $P^* = .99$, this method does not yield good results because tables we use are not sufficiently extensive.

Table

P^*	k	N	b	P_1^*
.90	3	16	.572	.90
	4	16	.543	.90
		25	.614	.90
.95	3	16	.505	.94
	4	16	.481	.94
		25	.558	.95
.99	4	16	.381	.98
		25	.465	.98

The procedure R_3 also has the monotonicity property like the procedure R_1. The proof of this is similar to the proof given in Section 2.9 and is omitted here.

Table I

Probability of Selecting a Subset Containing
the Best One of k Exponential Populations
Using Procedure R_1 (see Section 2.7)

k = 2

c	d = .1	d = .2	d = .3	d = .4	d = .5	d = .6	d = .7	d = .8	d = .9
.5	.99073	.95530	.89440	.74800	.73532	.73350	.65847	.60287	.56231
1.0	.99977	.99061	.95174	.89428	.87599	.73531	.65853	.62317	.56695
2.0	.99999	.99951	.99022	.96873	.93528	.85741	.73350	.65846	.57266
3.0	.99999	.99996	.99854	.98819	.96458	.89605	.81027	.71669	.57992
5.0	.99999	.99999	.99992	.99823	.98864	.94866	.86885	.76853	.60286
10.0	.99999	.99999	.99999	.99998	.99927	.98937	.94425	.84561	.67387

k = 3

c	d = .1	d = .2	d = .3	d = .4	d = .5	d = .6	d = .7	d = .8	d = .9
.5	.98199	.91768	.81708	.61149	.59375	.59240	.49979	.43822	.39577
1.0	.99955	.98176	.91155	.81696	.78949	.59374	.49982	.46024	.40053
2.0	.99999	.99903	.98104	.94166	.88443	.76272	.59240	.49979	.40643
3.0	.99999	.99994	.99712	.97736	.93471	.82100	.69506	.57195	.41397
5.0	.99999	.99999	.99986	.99653	.97828	.90755	.78077	.63882	.43822
10.0	.99999	.99999	.99999	.99996	.99856	.97981	.90097	.74886	.52013

k = 5

c	d = .1	d = .2	d = .3	d = .4	d = .5	d = .6	d = .7	d = .8	d = .9
.5	.96586	.85675	.70809	.46012	.43882	.43820	.34379	.28150	.24295
1.0	.99912	.96542	.84711	.70800	.67263	.43882	.34380	.30739	.25551
2.0	.99999	.99809	.96416	.89638	.80738	.63939	.43820	.34379	.26049
3.0	.99999	.99988	.99437	.95800	.88601	.71582	.55536	.41852	.26689
5.0	.99999	.99999	.99972	.99326	.95985	.84366	.66462	.49037	.28786
10.0	.99999	.99999	.99999	.99994	.99720	.96303	.83611	.62815	.36636

k = 10

c	d = .1	d = .2	d = .3	d = .4	d = .5	d = .6	d = .7	d = .8	d = .9
.5	.93146	.75121	.55273	.29648	.27666	.27652	.19938	.15887	.13403
1.0	.99805	.93058	.73734	.55267	.51440	.27666	.19938	.17265	.13673
2.0	.99999	.99579	.99579	.81398	.68549	.47916	.27652	.19938	.14009
3.0	.99999	.99975	.98795	.91861	.80021	.56601	.38890	.26275	.14444
5.0	.99999	.99999	.99939	.98573	.92266	.73802	.51054	.32605	.15887
10.0	.99999	.99999	.99999	.99986	.99406	.92983	.73530	.47373	.22001

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